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An explicit Hamiltonian formulation of surface waves in water of finite depth

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A variational formulation of water waves is developed, based on the Hamiltonian theory of surface waves. An exact and unified description of the two-dimensional problem in the vertical plane is obtained in the form of a Hamiltonian functional, expressed in terms of surface quantities as canonical variables. The stability of the corresponding canonical equations can be ensured by using positive definite approximate energy functionals. While preserving full linear dispersion, the method distinguishes between short-wave nonlinearity, allowing the description of Stokes waves in deep water, and long-wave nonlinearity, applying to long waves in shallow water. Both types of nonlinearity are found necessary to describe accurately large-amplitude solitary waves.

1. Introduction

When wind-generated sea waves approach shallow water, various physical processes influence the wave field, i.e. shoaling, refraction by depth and current variations, diffraction, nonlinear effects such as the transfer of energy between spectral components, and energy dissipation due to wave breaking and other mechanisms. In order to predict the transformation of wave properties in coastal areas, shallow-water wave models should be able to account for these effects. For that purpose, a number of model equations have been developed during the last two decades, largely based on linear wave theory (see Dingemans 1992 for a review).

However, because of the frequent occurrence of steep unsteady waves in nature, a nonlinear theory is needed which is capable of describing waves of limiting height, and is uniformly valid from deep to shallow water. In most existing wave theories it is assumed that the (steady) wave motion is irrotational, and the fluid inviscid and incompressible, which allows the introduction of a velocity potential. Apart from methods based on direct numerical solution of the governing exact equations (such as Fourier methods, cf. Schwartz & Fenton 1982), these theories divide into two main groups, namely Stokes wave theories and shallow-water wave theories (including cnoidal wave theory). Low-order Stokes theory is suitable to describe short waves, characterized by the condition that the ratio of the mean water depth to wavelength, h/L, is not much smaller than one, i.e. $h/L \geqslant 1/10$. Shallow-water theories are expected to be valid for longer waves, $h/L \ll 1$ (for details, see Dingemans 1992).

Several attempts to develop a *unified* theory of water waves have been made in the past, to combine deep- and shallow-water effects and obtain a greater range of validity. Whitham (1967, 1974) proposed an extension of the Korteweg–de Vries equation in the form of an integro-differential equation, which combines full linear dispersion with long-wave nonlinearity, and he showed that this equation exhibits the desired features of wave peaking and breaking in shallow water. Witting (1984) described an extension of Boussinesq-type theory with high-order dispersion, thus making the model capable of treating a wide variety of physical situations, e.g. high solitary waves and undular bores. Inspired by this work, Madsen, Murray & Sørensen (1991) further improved and extended the classical Boussinesq equations for two horizontal dimensions; numerical results were given for wave propagation and diffraction in relatively deep water. Yasuda, Ukai & Tsuchiya (1989) presented a Korteweg–de Vries-type model equation which rigorously satisfies the linear dispersion relation, and applies to swell with an arbitrary spectrum. Shields & Webster (1988) proposed a direct method, based on 'the theory of directed fluid sheets', in which the flow field is not irrotational; this leads to a hierarchy of approximate theories of increasing complexity, and solitary and periodic wave solutions show a significant improvement in comparison with the classical shallow-water approximations.

The present work is concerned with a Hamiltonian formulation of the water-wave problem. Zakharov (1968) was the first to note that the exact equations for wave motion in a perfect fluid constitute a dynamical system with a positive definite Hamiltonian functional (which represents the total energy of the fluid), i.e. the surface elevation and the velocity potential at the free surface are canonical variables in Hamilton's sense. This formalism was extended to arbitrary depth by Broer (1974) and Miles (1977). A systematic account of the symmetries and the corresponding conservation laws was given by Benjamin & Olver (1982); their results were deduced in a more simple way by Longuet-Higgins (1983). The theory is not necessarily restricted to irrotational flow or flow of uniform density (see Henyey 1983; Benjamin 1984; Lewis *et al.* 1986; Abarbanel, Brown & Yang 1988). A distinct advantage of the Hamiltonian formalism is that approximations which maintain the symmetries of the flow will automatically preserve the corresponding conservation laws. All complexities are contained in the calculation of the energy, and stability and bifurcation properties of water waves are related to the Hamiltonian structure of the problem and its symmetries (cf. Zufiria 1987; Saffman 1988). Further, any positive definite approximate Hamiltonian guarantees good dynamical behaviour of the corresponding canonical equations, i.e. stability of computer solutions for long periods of time (Broer 1974, 1975). The formalism is compatible with a deterministic as well as a stochastic description (Goldstein 1980; West *et al.* 1987, and references therein). Several equivalent formulations of the theory can be obtained by means of canonical transformations (Broer & Kobussen 1972; Milder 1977; Benjamin 1984).

A description of the canonical theorem for the complete water-wave problem is given in §2. The two-dimensional problem in the vertical plane is considered in §3; it leads to a unified Hamiltonian description of short and long waves and the corresponding nonlinearities, in water of arbitrary uniform depth. Special cases for gravity waves are dealt with in §4, namely deep water (Stokes waves) and shallow water (fairly long and solitary waves); it turns out that a clear distinction can be made between short-wave nonlinearity and long-wave nonlinearity. Finally, in §5 some applications and extensions of the present method are discussed.

2. Hamiltonian theory of surface waves

We begin with a description of the complete water-wave problem and its Hamiltonian formulation. Irrotational wave motion on the surface of an incompressible inviscid fluid of uniform density is considered. The velocity potential $\Phi(x,y,z,t)$ satisfies the Laplace equation

$$\nabla^2\Phi+\frac{\partial^2\Phi}{\partial z^2}=0, \tag{1}$$

with boundary conditions

$$\frac{\partial\Phi}{\partial z}+\nabla h\cdot\nabla\Phi=0 \quad \text{at} \quad z=-h(x,y), \tag{2}$$

and

$$\left.\begin{aligned}\frac{\partial\zeta}{\partial t}+\nabla\zeta\cdot\nabla\Phi&=\frac{\partial\Phi}{\partial z} && (3a)\\ \frac{\partial\Phi}{\partial t}+g\zeta-\tau\nabla\cdot\left[\frac{\nabla\zeta}{(1+(\nabla\zeta)^2)^{\frac{1}{2}}}\right]+\tfrac{1}{2}\left[(\nabla\Phi)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]&=0 && (3b)\end{aligned}\right\}\quad \text{at} \quad z=\zeta(x,y,t),$$

where $\zeta(x,y,t)$ is the surface elevation which is assumed to remain a single-valued function of horizontal position (cf. Benjamin & Olver 1982), $h(x,y)$ is the position of the (uneven) bottom, $\nabla\equiv(\partial/\partial x,\partial/\partial y)$ is the horizontal gradient-operator, g is the acceleration due to gravity, and τ the coefficient of surface tension. We note that the motion is fully determined by the surface elevation $\zeta(x,y,t)$ and the value of the velocity potential at the free surface,

$$\phi(x,y,t)=\Phi(x,y,\zeta(x,y,t),t). \tag{4}$$

When ζ and ϕ are known, Φ is given as the unique solution of the linear boundary-value problem (1), (2), and $\Phi=\phi$ at the boundary $z=\zeta$.

In the Hamiltonian theory of the problem (1)–(3), we consider the total energy of the fluid (omitting the uniform density as a factor) to be

$$\begin{aligned}\mathscr{H}&=\iint\mathrm{d}x\,\mathrm{d}y(V+T)\\ &=\iint\mathrm{d}x\,\mathrm{d}y\left\{\tfrac{1}{2}g\zeta^2+\tau[(1+(\nabla\zeta)^2)^{\frac{1}{2}}-1]+\frac{1}{2}\int_{-h}^{\zeta}\mathrm{d}z\left[(\nabla\Phi)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]\right\},\end{aligned} \tag{5}$$

where V is the potential energy density, T the kinetic energy density, and $H\equiv V+T$ is the Hamiltonian density.

The *canonical theorem* states that ζ and ϕ are canonical variables, with $\mathscr{H}(\zeta,\phi)$ as the corresponding Hamiltonian functional. The canonical equations

$$\frac{\partial\zeta}{\partial t}=\frac{\delta\mathscr{H}}{\delta\phi},\quad \frac{\partial\phi}{\partial t}=-\frac{\delta\mathscr{H}}{\delta\zeta}, \tag{6a, b}$$

where δ denotes the variational derivative, are equivalent to the exact free-surface boundary conditions (3), with the Laplace equation (1) and the bottom boundary condition (2) as constraints (cf. Zakharov 1968; Broer 1974; Broer, van Groesen & Timmers 1976; Miles 1977). Equivalent formulations of the theory can be obtained by means of canonical transformations (Broer & Kobussen 1972).

The essential difficulty in the application of (5), (6) is to find an explicit expression for the kinetic energy density

$$T=\frac{1}{2}\int_{-h}^{\zeta}\mathrm{d}z\left[(\nabla\Phi)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right], \tag{7}$$

as a functional of ζ and ϕ. The formal solution of this problem is discussed by Milder (1977), and approximate results have been obtained by several authors (see Miles 1981 for a review; Radder & Dingemans 1985; Neyzi & Nutku 1987; Creamer *et al.* 1989). To facilitate the analysis we use Green's theorem to write (7) in the form (Miles 1977):

$$T = \tfrac{1}{2}\phi \frac{\partial \zeta}{\partial t}. \tag{8}$$

Formally, it is possible to express $\partial\zeta/\partial t$ in terms of the surface quantities ζ and ϕ:

$$\frac{\partial \zeta}{\partial t} = \mathscr{F}(\zeta)\,\phi \tag{9}$$

where $\mathscr{F}$ is a non-local operator which is linear in its effect on ϕ but depends nonlinearly on ζ (cf. Milder 1977).

In the next section, the two-dimensional problem in the (x, z)-plane is considered, in which case analytic function theory can be used to simplify the analysis. For a horizontal bottom, an exact Hamiltonian formulation is given in which the vertical coordinate z is explicitly eliminated.

3. Explicit formulation of the two-dimensional problem

When the flow is two-dimensional, the wave motion can be described by the complex potential (cf. Lamb 1932, Art. 62)

$$F(Z,t) = \Phi(x,z,t) + \mathrm{i}\Psi(x,z,t), \tag{10}$$

where the complex function F is an analytic function of $Z = x + \mathrm{i}z$, and Ψ denotes the stream function. Using the Cauchy–Riemann relations for Φ and Ψ, we may write the kinematic free-surface condition ($3a$) in the form

$$\frac{\partial \zeta}{\partial t} = \left[-\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial z}\frac{\partial \zeta}{\partial x}\right]_{z=\zeta} = -\frac{\partial \psi}{\partial x}, \tag{11}$$

where ψ is defined as the stream function at the free surface,

$$\psi(x,t) = \Psi(x, \zeta(x,t), t). \tag{12}$$

(Note that (11) defines a local conservation law for the mass density ζ and the mass flux ψ, cf. Broer *et al.* 1976). From (8) and (11) the kinetic energy density becomes

$$T = -\tfrac{1}{2}\phi \frac{\partial \psi}{\partial x} \approx \frac{1}{2}\frac{\partial \phi}{\partial x}\psi, \tag{13}$$

where the second expression is a result of integration by parts. In order to express ψ in (13) in terms of the canonical variables ζ and ϕ, the fluid domain is mapped conformally into an infinite strip in the complex W-plane (cf. figure 1). The transformation is given by (cf. Woods 1961, §3.16)

$$Z(W) = \frac{1}{2}\int_{-\infty}^{\infty} \tanh\left[\tfrac{1}{2}\pi(W-\chi')\right]\zeta(\chi')\,\mathrm{d}\chi' + \frac{1}{2}\int_{-\infty}^{\infty} \coth\left[\tfrac{1}{2}\pi(W-\chi')\right]h(\chi')\,\mathrm{d}\chi' \tag{14}$$

(where the integrals are defined by $\int_{-\infty}^{\infty} \ldots = \lim_{A\to\infty}\int_{-A}^{A}\ldots$). The solution procedure to find Ψ at the surface consists of two steps: (i) solve the problem in the W-plane;

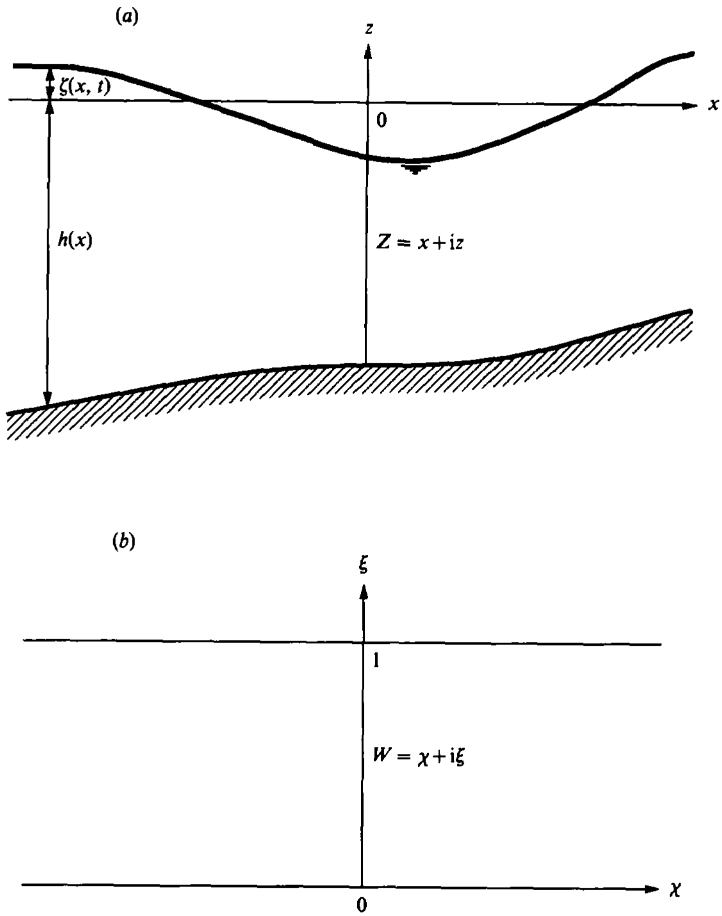

FIGURE 1. (*a*) Physical and (*b*) transformed coordinates.

(ii) find the inverse transformation $\chi(x)$ along the surface. Without loss of generality, we take $\Psi = 0$ at the bottom $z = -h$.

3.1. *Solution of the problem in the W-plane*

In the first step we solve the linear problem

$$\frac{\partial^2\Psi}{\partial\chi^2}+\frac{\partial^2\Psi}{\partial\xi^2} = 0, \quad 0 \leqslant \xi \leqslant 1, \tag{15}$$

$$\left.\begin{aligned} \Psi &= 0 && \text{at} \quad \xi = 0, \\ \Psi &= \psi(\chi) && \text{at} \quad \xi = 1, \end{aligned}\right\} \tag{16}$$

by means of Fourier transforms (cf. Byatt-Smith 1970; Davies 1985). If we take the Fourier transform of (15) with respect to the variable χ, we obtain

$$\frac{\mathrm{d}^2\hat{\Psi}}{\mathrm{d}\xi^2}-\kappa^2\hat{\Psi} = 0, \quad \hat{\Psi}(\kappa, 0) = 0, \tag{17}$$

with solution

$$\hat{\Psi}(\kappa, \xi) = A\sinh\kappa\xi = \frac{\tanh\kappa\xi}{\kappa}\frac{\partial\hat{\Psi}}{\partial\xi}. \tag{18}$$

Using the Cauchy–Riemann relation $\partial\Psi/\partial\xi = \partial\Phi/\partial\chi$ and the convolution theorem, we have

$$\Psi(\chi,\xi) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial\Phi}{\partial\chi'}\ln\tanh\left(\frac{\frac{1}{4}\pi|\chi-\chi'|}{\xi}\right)\mathrm{d}\chi', \tag{19}$$

which becomes, for $\xi = 1$,

$$\psi(\chi) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi}{\mathrm{d}\chi'}\ln\tanh\left(\tfrac{1}{4}\pi|\chi-\chi'|\right)\mathrm{d}\chi'. \tag{20}$$

To express (20) in terms of variables in the physical plane, we need an expression for the function $\chi(x)$ along the surface. This is accomplished in the second step.

3.2. *Solution of the inversion problem*

The imaginary part of the transformation (14) is given by (cf. Abramowitz & Stegun 1968, §4.5.51/52; Byatt-Smith 1971)

$$z(\chi,\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin\pi\xi}{\cosh\pi(\chi-\chi')+\cos\pi\xi}\zeta(\chi')\,\mathrm{d}\chi' - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin\pi\xi}{\cosh\pi(\chi-\chi')-\cos\pi\xi}h(\chi')\,\mathrm{d}\chi'. \tag{21}$$

The Fourier transform of (21) with respect to χ is

$$\hat{z}(\kappa,\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin\pi\xi\cos\kappa\chi'}{\cosh\pi\chi'+\cos\pi\xi}\mathrm{d}\chi'\,\hat{\zeta}(\kappa) - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin\pi\xi\cos\kappa\chi'}{\cosh\pi\chi'-\cos\pi\xi}\mathrm{d}\chi'\,\hat{h}(\kappa). \tag{22}$$

We can evaluate the integrals using (cf. Gradshteyn & Ryzhik 1980)

$$\int_{0}^{\infty}\frac{\cos px\,\mathrm{d}x}{\cosh\pi x+\cos q} = \frac{\sinh(pq/\pi)}{\sinh p\sin q}, \quad 0 \leqslant q < \pi, \tag{23}$$

to find

$$\hat{z}(\kappa,\xi) = \frac{\sinh\kappa\xi}{\sinh\kappa}\hat{\zeta}(\kappa) - \frac{\sinh\kappa(1-\xi)}{\sinh\kappa}\hat{h}(\kappa). \tag{24}$$

Differentiation with respect to ξ gives

$$\frac{\partial\hat{z}}{\partial\xi} = \frac{\kappa\cosh\kappa\xi}{\sinh\kappa}\hat{\zeta}(\kappa) + \frac{\kappa\cosh\kappa(1-\xi)}{\sinh\kappa}\hat{h}(\kappa). \tag{25}$$

Using the Cauchy–Riemann relation $\partial x/\partial\chi = \partial z/\partial\xi$ and the correspondence $\kappa \equiv \mathrm{i}\,\partial/\partial\chi$ we derive the symbolic operator equation, for $\xi \rightarrow 1$:

$$x(\chi) = [1/\tan(\mathrm{d}/\mathrm{d}\chi)]\,\zeta(\chi) + [1/\sin(\mathrm{d}/\mathrm{d}\chi)]\,h(\chi). \tag{26}$$

We are now faced with the problem of inverting the operator equation (26) to obtain the function $\chi(x)$.

In order to simplify the analysis, we assume henceforth that the bottom is horizontal, i.e. h is constant. Defining the local depth

$$\eta(\chi) = h + \zeta(\chi) \tag{27}$$

we obtain from (26)

$$\frac{\mathrm{d}x}{\mathrm{d}\chi} = \frac{\mathrm{d}}{\mathrm{d}\chi}\cot\left(\frac{\mathrm{d}}{\mathrm{d}\chi}\right)\eta. \tag{28}$$

An integral representation of (28) is given by

$$\frac{\mathrm{d}x}{\mathrm{d}\chi} = \eta(\chi) - \tfrac{1}{4}\pi \int_{-\infty}^{\infty} \frac{\eta(\chi-\chi')-\eta(\chi)}{\sinh^2 \frac{1}{2}\pi\chi'}\,\mathrm{d}\chi', \tag{29}$$

which can also be directly derived from (21) by taking the derivative with respect to ξ, in the limit of $\xi \rightarrow 1$. To solve the nonlinear integral equation (28), it needs to be expressed in terms of more simple operators, e.g. through a series expansion

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\cot\frac{\mathrm{d}}{\mathrm{d}\chi} = 1 - \frac{1}{3}\left(\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^2 - \frac{1}{45}\left(\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^4 - \dots. \tag{30}$$

This expansion is (formally) convergent for values of the argument $\|\mathrm{d}/\mathrm{d}\chi\| < \pi$, which is not suitable in case of deep water. Therefore we consider here an expansion in partial fractions (cf. Abramowitz & Stegun 1968, §4.3.91):

$$\frac{\mathrm{d}x}{\mathrm{d}\chi} = \eta + \sum_{m=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\chi}\left[\frac{1}{\mathrm{d}/\mathrm{d}\chi - m\pi} + \frac{1}{\mathrm{d}/\mathrm{d}\chi + m\pi}\right]\eta, \tag{31}$$

which is valid for all values of the argument. We introduce the variable p and the operator D

$$p = \int_0^x \frac{\mathrm{d}r}{\eta}, \quad \mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}p} = \eta\frac{\mathrm{d}}{\mathrm{d}x}, \tag{32}$$

and define the function $\epsilon(x)$ according to

$$\frac{\mathrm{d}x}{\mathrm{d}\chi} = (1+\epsilon)\,\eta. \tag{33}$$

Then we have $\mathrm{d}/\mathrm{d}\chi = (1+\epsilon)\,\mathrm{D}$, and the expansion (31) becomes

$$\epsilon = \frac{1}{\eta}(1+\epsilon)\sum_{m=1}^{\infty} \mathrm{D}\left[\frac{1}{(1+\epsilon)D - m\pi} + \frac{1}{(1+\epsilon)\,\mathrm{D} + m\pi}\right]\eta. \tag{34}$$

In order to solve this equation for the unknown function $\epsilon(x)$ we define the operators $\mathrm{G}_\epsilon^{(m)}$ and $\mathrm{G}_0^{(m)}$ by

$$\mathrm{G}_\epsilon^{(m)} \equiv \mathrm{D}\frac{1}{(1+\epsilon)\,\mathrm{D} - m\pi}; \quad \mathrm{G}_0^{(m)} \equiv \frac{\mathrm{D}}{\mathrm{D} - m\pi}, \tag{35}$$

and try to express $\mathrm{G}_\epsilon^{(m)}$ in terms of $\mathrm{G}_0^{(m)}$. We have the operator identity†

$$\mathrm{D}\frac{1}{(1+\epsilon)\,\mathrm{D} - m\pi} \equiv \frac{\mathrm{D}}{\mathrm{D} - m\pi}[(1+\epsilon)\,\mathrm{D} - m\pi - \epsilon\mathrm{D}]\frac{1}{(1+\epsilon)\,\mathrm{D} - m\pi}, \tag{36}$$

hence

$$\mathrm{G}_\epsilon^{(m)} = \mathrm{G}_0^{(m)}(1 - \epsilon\mathrm{G}_\epsilon^{(m)}). \tag{37}$$

This is an integral equation in $\mathrm{G}_\epsilon^{(m)}$, which can be solved by iteration to yield

$$\mathrm{G}_\epsilon^{(m)} = \sum_{\lambda=0}^{\infty} (-1)^\lambda\,\mathrm{G}_0^{(m)}(\epsilon\mathrm{G}_0^{(m)})^\lambda. \tag{38}$$

Substituting (38) into (34):

$$\epsilon = (1+\epsilon)\sum_{\lambda=0}^{\infty} (-1)^\lambda I_\lambda, \tag{39}$$

† In quantum mechanics, a similar technique has been employed in the method of stationary perturbations (cf. Messiah 1969, Chap. XVI, Sec. III).

with
$$I_\lambda = \frac{1}{\eta} \sum_{m=1}^{\infty} [\mathrm{G}_0^{(m)}(\epsilon \mathrm{G}_0^{(m)})^\lambda + \mathrm{G}_0^{(-m)}(\epsilon \mathrm{G}_0^{(-m)})^\lambda]\,\eta, \tag{40}$$

we are able to solve (34) by iteration:

$$\epsilon_0 = 0; \quad \epsilon_1 = I_0/(1-I_0); \quad \epsilon_2 = (I_0 - I_1)/(1-I_0+I_1); \ldots. \tag{41}$$

To evaluate the integrals I_λ we need an integral representation of the operators $\mathrm{G}_0^{(m)}$ and $\mathrm{G}_0^{(-m)}$ in (40). The result is (cf. appendix A for the details):

$$I_0 = -\frac{1}{\eta(p)} \int_0^\infty \frac{\mathrm{d}q}{\exp \pi q - 1} \frac{\mathrm{d}}{\mathrm{d}q} [\eta(p+q) + \eta(p-q)];$$

$$I_1 = \frac{1}{\eta(p)} \int_0^\infty \int_0^\infty \frac{\mathrm{d}q\, dq'}{\exp \pi (q+q') - 1} \frac{\mathrm{d}}{\mathrm{d}q}$$
$$\times \left[\epsilon(p+q) \frac{\mathrm{d}}{\mathrm{d}q'} \eta(p+q+q') + \epsilon(p-q) \frac{\mathrm{d}}{\mathrm{d}q'} \eta(p-q-q') \right], \tag{42}$$

in which ϵ should be approximated by ϵ_1; ... and so on.

In case of finite-depth water, we have $\|\mathrm{D}\| < \pi$, and we may expand $\mathrm{G}_0^{(m)}$ in a series

$$\mathrm{G}_0^{(m)} = -\frac{\mathrm{D}}{m\pi} \left[1 + \frac{\mathrm{D}}{m\pi} + \left(\frac{\mathrm{D}}{m\pi}\right)^2 + \left(\frac{\mathrm{D}}{m\pi}\right)^3 + \ldots \right], \tag{43}$$

and likewise $\mathrm{G}_0^{(-m)}$. To express the expansion (39) in this case in a more convenient form we introduce the functions F_k, $k = 0, 1, 2 \ldots$ as follows:

$$F_k = \frac{1}{\eta} \mathrm{D}^k \eta; \quad F_0 = 1. \tag{44}$$

From (32) and (44) we have the recursion relations

$$\mathrm{D}(\eta F_k) = \eta F_{k+1}, \tag{45a}$$

and
$$\mathrm{D} F_k = F_{k+1} - F_k F_1. \tag{45b}$$

On substituting (43) into (39), (40) and using the relations (45) for F_k, it follows that ϵ can be expanded in terms of these functions F_k; up to fourth order in η we get

$$\epsilon = -\tfrac{1}{3}F_2 - \tfrac{1}{45}(F_4 - 5F_3 F_1 - 10F_2^2 + 5F_2 F_1^2) - \ldots. \tag{46}$$

This result can also be deduced from the expansion (30), and a representation of (46) in terms of η and its derivatives is given in appendix B. From this, a similar expansion of the expression $1/(1+\epsilon)$ can be obtained easily.

3.3. *Discussion of the results*

To sum up, from (13), (20), (33) and (46) we have the following expression for the kinetic energy density:

$$T = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x'\, \phi_x \phi_{x'} R_\epsilon(x, x'; \eta), \tag{47a}$$

where the *symmetric* function R_ϵ is given by

$$R_\epsilon(x, x'; \eta) \equiv -\frac{1}{\pi} \ln \tanh \frac{\pi}{4} \left| \int_x^{x'} \frac{\mathrm{d}r}{(1+\epsilon)\,\eta} \right|. \tag{47b}$$

We note that (47a) can be written in the form (8), (9) after integration by parts.

Before presenting an analysis of some special cases, we shall first consider a useful property of the present formulation. We infer from (13) and (20) that the kinetic energy functional, as expressed in the χ-variable, is *positive definite*, i.e. its spectrum function is positive and bounded (cf. Broer 1974). This property is preserved on transforming from the χ-variable to the x-variable provided the Jacobian (33) does not change sign; a sufficient condition is

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} > 0, \text{ and bounded,} \tag{48}$$

except perhaps at isolated points. This condition is valid as long as the bottom is well covered (i.e. $\eta > 0$) and wave breaking does not occur (i.e. $1/(1+\epsilon) > 0$). Consequently, *any* truncated expansion of the function ϵ, whether given by (41) or (46), leads to stable model equations when (48) is satisfied, in view of the positive definite character of the corresponding approximate Hamiltonian functional.

This is a distinct advantage over the classical methods such as the Stokes expansion, and other expansions reported in the recent literature. For instance, Milder (1990) has made a perturbation analysis of the alternative approach of West *et al.* (1987), which is based on an expansion of the Hamiltonian in a series in surface slope; he found an exponentially growing instability for high wavenumbers when the series is truncated at low order. While this unphysical behaviour can be partly remedied by adding terms of higher order, it would not occur if it were possible to keep the approximate Hamiltonians positive definite. To that end, the present method is applicable, although it requires the numerical solution of integral (or integro-differential) equations.

We now turn to some limiting cases, which correspond to the classical theories of Stokes and Boussinesq, without giving a rigorous asymptotic analysis. We will consider only gravity waves, neglecting the effects of surface tension.

4. Special cases

The formulation (47) is exact for two-dimensional wave motion in water of constant finite depth h. Consequently, both Stokes theory and Boussinesq theory are included as special cases. Almost trivially, the linear wave theory results by taking the surface elevation ζ infinitely small, so that in (47) $\epsilon = 0$ and $\eta = h$; then R_ϵ becomes

$$R_0(x, x'; h) = -\frac{1}{\pi} \ln \tanh \frac{\pi}{4h} |x'-x|, \tag{49}$$

which is valid for low waves (cf. Broer 1974).

In more general cases, the behaviour will depend on the rate of convergence of the expansion (39), which is given by $\|\epsilon \mathrm{G}_0^{(m)}\| < 1$. First, let us consider weakly nonlinear waves. For deep water it turns out that this amounts to $\epsilon = O(\kappa a)$, where κa denotes the wave steepness (κ is the wavenumber, a the wave amplitude). Then Stokes theory is appropriate, $\eta \approx h$, and we have for R_ϵ:

$$R_\epsilon(x, x'; h) = -\frac{1}{\pi} \ln \tanh \frac{\pi}{4h} \left| \int_x^{x'} \frac{\mathrm{d}r}{1+\epsilon} \right|. \tag{50}$$

When the depth is finite, the expression (46) indicates that the expansion (30) is valid as long as $\kappa h < \pi$, i.e. up to fairly deep water. For shallow water we have $\epsilon = O(\kappa^2 ha)$, and Boussinesq theory is appropriate. Then we put $\epsilon \approx 0$, and we get for R_ϵ:

$$R_0(x, x'; \eta) = -\frac{1}{\pi} \ln \tanh \frac{\pi}{4h} \left| \int_x^{x'} \frac{\mathrm{d}r}{1+\zeta/h} \right|. \tag{51}$$

Thus the function $\epsilon(x)$ represents short-wave nonlinearity, while the relation $\zeta(x)/h$ accounts for long-wave nonlinearity. In case of strongly nonlinear waves, e.g. large-amplitude solitary waves, we may expect that $\epsilon = O(1)$. In particular, the expression $1/(1+\epsilon)$ becomes zero at a stagnation point.

In the following paragraphs we will consider these cases in more detail.

4.1. *Stokes waves*

Restricting the analysis to deep water, $\kappa h \rightarrow \infty$, we look for periodic solutions with a steady profile, i.e. ζ and ϕ being periodic in the phase function $\theta(x,t)$, with wavenumber $\kappa = \partial\theta/\partial x$ and frequency $\omega = -\partial\theta/\partial t$, both assumed to be constant. The Stokes expansion is taken as

$$\zeta(\theta) = a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + O(a_1^4), \tag{52a}$$

$$\phi(\theta) = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + O(b_1^4). \tag{52b}$$

In order to find the coefficients a_n and b_n, we apply the 'average variational principle' of Whitham (1974, §16.6). The Lagrangian density is in this case given by

$$L = \zeta\phi_t + \tfrac{1}{2}g\zeta^2 + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x'\, \phi_x \phi_{x'} R_\epsilon(x, x'; h), \tag{53}$$

where $R_\epsilon(x, x'; h)$ is defined by (50). We propose to find the Stokes expansion (52) up to the third order inclusive; so we need to compute L to the sixth order. To simplify the necessary integrations, we use a property of Dirichlet integrals:

$$\lim_{h\rightarrow\infty} \int_0^\infty f(u) \frac{\sin hu}{u} \mathrm{d}u = \lim_{h\rightarrow\infty} \int_0^\infty f\left(\frac{v}{h}\right) \frac{\sin v}{v} \mathrm{d}v = \tfrac{1}{2}\pi f(0^+), \tag{54}$$

where $f(0^+) = \lim_{u\rightarrow 0} f(u)$, and $f(u)$ satisfies Dirichlet's conditions (cf. Davies 1985).

Then we obtain for the integrand in (50), as is shown in Appendix C, the expression:

$$1/(1+\epsilon) = 1 - \gamma_1 \cos\theta - 2\gamma_2 \cos 2\theta - 3\gamma_3 \cos 3\theta + O(a_1^4), \tag{55}$$

with

$$\gamma_1 = \kappa a_1(1 - \kappa a_2 - \tfrac{1}{4}\kappa^2 a_1^2), \quad \gamma_2 = \kappa a_2 - \kappa a_1(\kappa a_3 + \tfrac{1}{2}\kappa^2 a_1 a_2 + \tfrac{1}{24}\kappa^3 a_1^3), \quad \gamma_3 = \kappa a_3.$$

(For steady wave motion we may choose the velocity potential ϕ as χ-variable, and we have, apart from a factor κ^2/ω: $\gamma_n \sim b_n$, $n = 1, 2, 3, \ldots$. This can serve as a check on the calculations.)

Defining the new variables $u = \pi(x'-x)/(4h)$ and $v = 2\kappa hu/\pi$, it follows from (50) and (55) that

$$R_\epsilon = -\frac{1}{\pi} \ln \tanh\left[|u|\,(1+\delta(\theta, v))\right], \tag{56}$$

with

$$\delta(\theta, v) = -\frac{1}{v}\left[\gamma_1 \sin v \cos(\theta+v) + \gamma_2 \sin 2v \cos 2(\theta+v) + \gamma_3 \sin 3v \cos 3(\theta+v)\right]. \tag{57}$$

Expanding R_ϵ in a power series in δ, and using the property (54) we find for the kinetic energy density, in the limit of $\kappa h \to \infty$:

$$T = -\frac{1}{\pi}\kappa\phi_\theta \int_{-\infty}^{\infty} \mathrm{d}v\, \phi_{\theta'}[\ln\tanh|u| + \delta - \tfrac{1}{2}\delta^2 + \tfrac{1}{3}\delta^3 - \tfrac{1}{4}\delta^4], \tag{58}$$

in which $\theta' = \theta + 2v$.

With the help of the integrals given in Appendix D, using the rules for products of sines and cosines, and taking the average over one period in θ, the mean Lagrangian becomes

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{2}\omega(a_1 b_1 + 2a_2 b_2 + 3a_3 b_3) + \tfrac{1}{4}g(a_1^2 + a_2^2 + a_3^2)\\ &+ \tfrac{1}{4}\kappa b_1^2[1 + \kappa a_2 + \tfrac{1}{4}\kappa^2 a_1^2(1 - \tfrac{1}{8}\kappa^2 \alpha_1^2) + \tfrac{1}{2}\kappa^2 a_2^2]\\ &+ \kappa b_1(b_2 - \tfrac{1}{8}\kappa a_1 b_1)(\kappa a_3 + \tfrac{1}{2}\kappa^2 a_1 a_2 + \tfrac{1}{24}\kappa^3 a_1^3) + \tfrac{1}{2}\kappa b_2^2 + \tfrac{3}{4}\kappa b_3^2.\end{aligned} \tag{59}$$

Finally, the coefficients a_n and b_n can be found by variation of $\mathscr{L}$, which yields, to third order in κa_1:

$$\left.\begin{aligned} &b_3 = \frac{\omega}{\kappa}a_3; \quad a_3 = \tfrac{3}{8}\kappa^2 a_1^3; \quad b_2 = \frac{\omega}{\kappa}a_2; \quad a_2 = \tfrac{1}{2}\kappa a_1^2;\\ &b_1 = \frac{\omega}{\kappa}a_1(1 - \tfrac{3}{4}\kappa^2 a_1^2); \quad \omega^2 = g\kappa(1 + \kappa^2 a_1^2).\end{aligned}\right\} \tag{60}$$

The last expression in (60) is the Stokes dispersion relation for gravity waves.

Thus we have shown that the present formulation, as it is given by (53), provides a correct description of third-order Stokes waves in deep water (see e.g. Fenton 1985).

4.2. *Fairly long, fairly low waves*

In this case we are concerned with approximations which are valid when the parameters $\mu = \kappa h$ and $\alpha = a/h$ are small but finite, $O(\mu^2) = O(\alpha) \leqslant 1$. This implies that the effects of (long-wave) nonlinearity α and dispersion μ^2 are of the same order (Boussinesq theory). Broer (1974, 1975) used the Hamiltonian formalism to obtain stable evolution equations of Boussinesq type; these equations can be derived from a Hamiltonian density of the form (cf. Broer 1975, equation 2.6):

$$H_\mathrm{B} = \tfrac{1}{2}g\zeta^2 + \tfrac{1}{2}\zeta\phi_x^2 + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x'\phi_x\phi_{x'}R_0(x,x';h), \tag{61}$$

where $R_0(x,x';h)$ is defined by (49).

We will show here that (61) can be considered as a special case of the present formulation

$$H = \tfrac{1}{2}g\zeta^2 + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x'\phi_x\phi_{x'}R_0(x,x';\eta), \tag{62}$$

where $R_0(x,x';\eta)$ is given by (51), and $\eta = h + \zeta$. For that purpose, we scale the variables in the usual way, as follows (cf. Mei 1989, Chap. 11):

$$\tilde{x} = \kappa x, \quad \tilde{\zeta} = \frac{\zeta}{a}, \quad \tilde{\phi} = \frac{\kappa\phi}{\alpha(gh)^{\frac{1}{2}}}, \quad \tilde{H} = \frac{H}{ga^2}. \tag{63}$$

Omitting the tildes, we obtain from (62)

$$H = \tfrac{1}{2}\zeta^2 + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x' \phi_x \phi_{x'} R_0(x, x'; \zeta, \alpha, \mu), \tag{64}$$

with

$$R_0(x, x'; \zeta, \alpha, \mu) = -\frac{1}{\pi\mu} \ln \tanh \frac{\pi}{4\mu} \left| \int_x^{x'} \frac{\mathrm{d}r}{1+\alpha\zeta} \right|. \tag{65}$$

Expanding (65) in a power series in α, we have

$$R_0(x, x'; \zeta, \alpha, \mu) = -\frac{1}{\pi\mu} \ln \tanh \frac{\pi}{4\mu} |x'-x| + \alpha \frac{f(x'-x, \mu)}{x'-x} \int_x^{x'} \zeta \, \mathrm{d}r + O(\alpha^2), \tag{66}$$

where

$$f(z, \mu) = \frac{1}{2\mu^2} \frac{|z|}{\sinh(\pi/2\mu)|z|}, \quad f(0, \mu) = \frac{1}{\pi\mu}. \tag{67}$$

As $f(z, \mu)$ is normalized to unity,

$$\int_{-\infty}^{\infty} f(z, \mu) \, \mathrm{d}z = 1, \tag{68}$$

it defines the delta function when μ tends to zero (cf. Lighthill 1962):

$$f(0, \mu) \to \infty; \quad f(z, \mu) \to 0, \quad z \neq 0. \tag{69}$$

Consequently, for μ and α sufficiently small, (64) becomes

$$H = \tfrac{1}{2}\zeta^2 + \tfrac{1}{2}\alpha\zeta\phi_x^2 + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x' \phi_x \phi_{x'} R_0(x, x'; 0, 0, \mu), \tag{70}$$

which is identical to (61) in physical variables.

From (70) it is seen that in Broer's Hamiltonian (61) nonlinearity and dispersion appear in separate terms, which facilitates further analysis and approximations. Katopodes & Dingemans (1989), using model equations based on (61), performed some simple numerical tests, which demonstrate a remarkable improvement in short-wave stability over the classical Boussinesq-type equations. On the other hand, stability is not guaranteed for waves of large amplitude, because the Hamiltonian (61) need not be positive definite in that case. For a more complete treatment of this problem, we refer to Broer *et al.* (1976).

4.3. *The solitary wave*

We now turn to the limiting case of the solitary wave of finite amplitude, when the parameter α is not assumed to be small. This problem has been the subject of extensive investigations (for a review, see Miles 1980, and Schwartz & Fenton 1982 for further discussion). Being of permanent form, the solitary wave exhibits a balance between nonlinearity and dispersion, which is preserved up to the highest wave. We consider the Hamiltonian (62), which we repeat here for convenience,

$$H = \tfrac{1}{2}g\zeta^2 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x' \phi_x \phi_{x'} \ln \tanh \frac{\pi}{4} \left| \int_x^{x'} \mathrm{d}r/\eta \right|, \quad \eta = h + \zeta, \tag{71}$$

and the question arises whether (71) is suitable to describe this case. Actually, this form combines full linear dispersion with long-wave nonlinearity, retaining terms of all orders in μ and α. On the other hand, short-wave nonlinearity may possibly play a part when $\epsilon = O(1)$. In particular, it is of interest to know to what extent the formulation (71) can describe the highest solitary wave with a sharp crest, enclosing an angle of 120°, and a maximum value of $\alpha, \alpha_{\max} \approx 5/6 = 0.833$. For that purpose we need to study the corresponding canonical equations. In Appendix E it is shown that these equations are given by

$$\frac{\partial \zeta}{\partial t} = \frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}x'\phi_{x'} \ln \tanh \frac{\pi}{4}\left|\int_{x}^{x'} \mathrm{d}r/\eta\right|, \tag{72a}$$

$$\frac{\partial \phi}{\partial t} = -g\zeta - \frac{1}{2\eta^2}\int_{-\infty}^{x} \mathrm{d}x' \int_{x}^{\infty} \mathrm{d}x'' \frac{\phi_{x'}\phi_{x''}}{\sinh\frac{\pi}{2}\left|\int_{x'}^{x''}\mathrm{d}r/\eta\right|}. \tag{72b}$$

The analysis of these nonlinear integral equations is, however, difficult and it is therefore appropriate to treat here first the more simple equations, which can be derived from Broer's Hamiltonian (61):

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}(\zeta\phi_x) - \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}x'\phi_{x'} R_0(x, x'; h), \tag{73a}$$

$$\frac{\partial \phi}{\partial t} = -g\zeta - \tfrac{1}{2}\phi_x^2, \tag{73b}$$

where $R_0(x, x'; h)$ is given by (49).

While (73*a*, *b*) are valid for small values of the parameters μ and α, it can be shown that equations of this form have solutions which peak at a critical height. We consider steady solitary-wave solutions in a reference frame moving with the wave speed c,

$$\zeta(x,t) = \zeta(X), \quad \phi(x,t) = \phi(X) + cX, \quad X = x - ct, \tag{74}$$

and we let $\zeta \to 0$ and $\phi_X \to -c$ for $X \to \pm\infty$.

We normalize the variables according to

$$\mathsf{X} = X/h, \quad \mathsf{Z} = \zeta/h, \quad \mathsf{V} = 1 + \phi_X/c, \quad \mathsf{C} = c/(gh)^{\frac{1}{2}}, \tag{75}$$

where C denotes the Froude number.

Substituting (74) and (75) into (73), integrating once and taking into account the conditions at infinity, we obtain an integral equation for the velocity $\mathsf{V}(\mathsf{X})$,

$$\mathsf{Z}(\mathsf{X})\,[1 - \mathsf{V}(\mathsf{X})] = \int_{-\infty}^{\infty} \mathrm{d}\mathsf{Y}\, \mathsf{V}(\mathsf{Y})\, R_0(\mathsf{X} - \mathsf{Y}), \tag{76a}$$

with

$$\mathsf{Z} = \tfrac{1}{2}\mathsf{C}^2\mathsf{V}(2 - \mathsf{V}). \tag{76b}$$

The kernel in (76*a*) is normalized to unity,

$$\int_{-\infty}^{\infty} \mathrm{d}\mathsf{Y}\, R_0(\mathsf{X} - \mathsf{Y}) = 1, \tag{77}$$

for all values of the variable X. We look for solutions $\mathsf{V}(\mathsf{X})$ of (76), which are symmetrical with respect to the origin, with $0 \leqslant \mathsf{V}(\mathsf{X}) \leqslant 1$. Apart from the trivial solution $\mathsf{V}_0 \equiv 0$, we have solutions of uniform flow,

$$\mathsf{V}_1 \equiv \tfrac{1}{2}[3 - (1 + 8/\mathsf{C}^2)^{\frac{1}{2}}], \quad \mathsf{C} \geqslant 1, \tag{78}$$

and the solitary wave *bifurcates* from the case $\mathsf{C} = 1$.

It follows from (76*b*) that the highest-possible wave occurs when $V=1$, i.e. when the crest has a stagnation point, and the amplitude has a maximum value $\alpha_{\max} \equiv Z_{\max} = \frac{1}{2}C^2$. This is a distinct quality of the limiting solitary wave. To see whether this phenomenon actually appears here, we differentiate (76*a*) with respect to X to yield

$$\tfrac{3}{2}C^2(1+\tfrac{1}{3}\sqrt{3}-V)(1-\tfrac{1}{3}\sqrt{3}-V)\frac{\mathrm{d}V}{\mathrm{d}X} = \int_{-\infty}^{\infty} \mathrm{d}Y \frac{\mathrm{d}V}{\mathrm{d}Y} R_0(X-Y). \tag{79}$$

This equation is similar to the integral equation of Whitham (1967; 1974, sec. 13.14), who showed that an equation of this type can describe periodic and solitary waves with the desired peaking. Here, the derivative $\mathrm{d}V/\mathrm{d}X$ becomes discontinuous at $V_{\max} = 1-\frac{1}{3}\sqrt{3}$, where $Z_{\max} = \frac{1}{3}C^2$ according to (76*b*). From (76*a*) and (77) we have $Z_{\max}(1-V_{\max}) < V_{\max}$, and we obtain an upper bound for the amplitude

$$Z_{\max} < \sqrt{3}-1 = 0.732. \tag{80}$$

We have thus found that the solution of (73), while giving a qualitatively correct description of the solitary wave of moderate amplitude, does not tend to the limiting-wave solution with a stagnation point at the crest, as the Froude number increases. On the other hand, the preceding analysis suggests that the canonical equations (72) will provide a better description of this quality, and one might even conjecture that the Hamiltonian system (71) constitutes an exact model for the (un)steady solitary wave, up to the point of breaking. However, Zwartkruis (1991) transformed the equations (72) for the steady solitary wave to an integral equation of Hammerstein type for the surface profile Z; he solved this equation numerically and found solutions for values of the Froude number C as large as 5, and presumably there exist solutions for *all* values of $C \geqslant 1$.

We may conclude that in order to describe accurately surface waves of maximum height, it is necessary to take into account short-wave nonlinearity, which was neglected in the formulation (71). To this end, we shall derive an exact integral equation and obtain a numerical solution for the solitary wave of maximum amplitude. From (6*a*) and (47*a*), we have

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}x'\, \phi_{x'} R_\epsilon(x, x'; \eta), \tag{81a}$$

where the kernel is given by (47*b*).

An equivalent of (6*b*) can be found by differentiating (4) with respect to x and t, and eliminating Φ_x, Φ_z and Φ_t, taken at the surface, from the free-surface boundary conditions (3*a*, *b*). The result is

$$\partial\phi/\partial t = -g\zeta - \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}(\zeta_t + \phi_x \zeta_x)^2/(1+\zeta_x^2). \tag{81b}$$

The system of equations (81*a*, *b*) thus represents a Hamiltonian system, which is exact for surface waves of arbitrary height. Assuming steady wave motion and taking coordinate axes moving with the wave speed c, we can replace the (χ, ξ) coordinates in figure 1 by $-(\Phi, \Psi)/hc$ coordinates. Consequently,

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} = -\frac{1}{hc}\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{1}{(1+\epsilon)\,\eta}, \tag{82}$$

according to (33).

Changing to non-dimensional variables as before, we obtain from (81) and (82) an integral equation for the surface profile $Z(X)$,

$$Z(X) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}Y[1-W(Y)]\ln\tanh\frac{\pi}{4}\left|\int_X^Y W(X')\,\mathrm{d}X'\right|, \tag{83a}$$

with $$W(X) = [(1-Z(X)/Z_{\max})(1+(\mathrm{d}Z/\mathrm{d}X)^2)]^{\frac{1}{2}}, \quad Z_{\max} = \tfrac{1}{2}C^2. \tag{83b}$$

This equation is equivalent to the integral equation of Byatt-Smith (1970) for the case of a non-periodic wave. His equation is expressed in terms of the velocity potential as independent variable, and exhibits a strong singularity in the integrand at the stagnation point. The advantage of (83) is that it is quasi-regular, while the wave profile is given explicitly by the solution $Z(X)$. In order to find the value of the maximum amplitude $Z_{\max}$, we try to model the discontinuity in surface slope at the crest by writing (cf. Longuet-Higgins 1974; Pennell & Su 1984)

$$Z(X) = \sum_{n=1}^{N} A_n \,\mathrm{e}^{-n|X|}, \quad Z_{\max} = Z(0). \tag{84}$$

Herein, the N unknowns A_n have to be found by satisfying (83) at the N mesh points

$$X_n = \ln(N/n), \quad n = 1, \ldots, N. \tag{85}$$

The resulting system of N nonlinear equations is solved by Newton iterations. A fairly accurate solution (with an overall residual error less than 0.2% of $Z_{\max}$) is found for $N = 3$:

$$A_1 = 1.190, \quad A_2 = -0.460, \quad A_3 = 0.104, \tag{86}$$

which gives a maximum amplitude $Z_{\max} = 0.834$ and an interior angle at the crest of 119.5°. For larger values of N, the method becomes unstable and an alternative method should be used (see e.g. Baker 1977, Chap. 5; Baker & Miller 1982).

5. Discussion and conclusions

We have developed a variational formulation of water waves in the form of a Hamiltonian functional, providing an exact description of two-dimensional wave motion in water of constant finite depth h. An explicit expression for the kinetic energy density in terms of the surface quantities η and ϕ is given by (47); the corresponding (approximate) kinetic energy functional is positive definite on the condition (48), which guarantees stability of the resulting canonical equations with respect to short waves. Dependent on the behaviour of the local depth $\eta = h+\zeta$ and its derivatives, we can discern short-wave nonlinearity, represented by the function $\epsilon(x)$, and long-wave nonlinearity, on account of the relation $\zeta(x)/h$. For weakly nonlinear waves, it is found that $\epsilon = O(\kappa a)$ in deep water (Stokes theory), and $\epsilon = O(\kappa^2 ha) \approx 0$ in shallow water (Boussinesq theory). For strongly nonlinear waves, $\epsilon = O(1)$ and both types of nonlinearity should be included in an accurate description of high solitary waves. In all cases, full linear dispersion is retained.

In a similar way, we can treat the case of an *uneven bottom* by an expansion of the operator $1/\sin(\mathrm{d}/\mathrm{d}\chi)$ in (26) in partial fractions, analogous to (31). As a first approximation, we can just replace h with $h(x)$ in the functional involved, for waves in water of slowly varying depth. Using this approximation in (71), Zwartkruis (1991) performed some preliminary numerical experiments, to study the evolution of a solitary wave passing over a submerged obstacle. While this problem needs further study, the results already indicate the robustness of the present method, even with simple numerical means.

Finally, we mention some other applications and extensions of the method. Wave steepening of long waves, leading to the development of an undular or turbulent bore, should be considered (see Peregrine 1985 for a description of this phenomenon). In deeper water, the evolution of the spectrum of waves propagating over an uneven bottom deserves attention. A numerical approach may be developed through an expansion of the wave field in terms of *orthonormal* functions $S_k(x)$,

$$\zeta(x,t) = \sum_k q_k(t) S_k(x), \quad \phi(x,t) = \sum_k p_k(t) S_k(x), \tag{87}$$

so that the coefficients also obey canonical equations:

$$\dot{q}_k = \frac{\partial \mathscr{H}}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial \mathscr{H}}{\partial q_k}. \tag{88}$$

A Fourier expansion leads to a spectral description of the wave motion (Miles 1977; West *et al.* 1987). Although extension of the theory to two horizontal dimensions is desirable, it is not trivial and needs further research. Of even more importance is the possibility of including dissipative effects in the variational formulation (see e.g. Anthony 1987; Salmon 1988; Vujanovic & Jones 1989). Investigations along these lines are awaited which allow for wave breaking and other irreversible processes.

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Appendix A

Let

$$\mathrm{G}_0^{(m)} \eta \equiv \frac{\mathrm{D}}{\mathrm{D}-m\pi}\eta = g_m, \quad m = 1,2,3,\ldots, \tag{A 1}$$

then

$$\mathrm{D}g_m - m\pi g_m = \mathrm{D}\eta. \tag{A 2}$$

With $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}p$, the solution of the linear first-order differential equation (A 2) is given by

$$g_m(p) = \mathrm{e}^{m\pi p}\left[\int \mathrm{e}^{-m\pi p'} \frac{\mathrm{d}\eta}{\mathrm{d}p'} \mathrm{d}p' + \text{const.}\right]. \tag{A 3}$$

The integration constant is fixed by the condition that $g_m(p)$ remains finite when p tends to $\pm\infty$. Then it follows from (A 1)–(A 3) that

$$\mathrm{G}_0^{(m)} = -\int_p^\infty \mathrm{d}p'\, \mathrm{e}^{m\pi(p-p')} \frac{\mathrm{d}}{\mathrm{d}p'}, \tag{A 4}$$

and by a similar argument

$$\mathrm{G}_0^{(-m)} = \int_{-\infty}^p \mathrm{d}p'\, \mathrm{e}^{-m\pi(p-p')} \frac{\mathrm{d}}{\mathrm{d}p'}, \tag{A 5}$$

where m is a positive integer. If we set $p-p' = -q$, resp. $p-p' = q$, then (A 4) and (A 5) become

$$\mathrm{G}_0^{(m)} = -\int_0^\infty \mathrm{d}q\, \mathrm{e}^{-m\pi q} \frac{\mathrm{d}}{\mathrm{d}q\,|_{p+q}}, \quad \mathrm{G}_0^{(-m)} = -\int_0^\infty \mathrm{d}q\, \mathrm{e}^{-m\pi q} \frac{\mathrm{d}}{\mathrm{d}q\,|_{p-q}}, \quad \text{(A 6), (A 7)}$$

where it is understood that the derivative under the integral has to be evaluated for $p' = p+q$, resp. $p' = p-q$.

The expression for I_λ follows from a repeated application of (A 6) and (A 7), and summation of the resulting geometric series, in the expansion (40):

$$I_0 = -\frac{1}{\eta(p)}\int_0^\infty \frac{\mathrm{d}q}{\exp \pi q - 1}\frac{\mathrm{d}}{\mathrm{d}q}[\eta(p+q)+\eta(p-q)]; \tag{A 8}$$

$$I_1 = \frac{1}{\eta(p)}\int_0^\infty\int_0^\infty \frac{\mathrm{d}q\,\mathrm{d}q'}{\exp \pi(q+q')-1}\frac{\mathrm{d}}{\mathrm{d}q}\left[\epsilon(p+q)\frac{\mathrm{d}}{\mathrm{d}q'}\eta(p+q+q') + \epsilon(p-q)\frac{\mathrm{d}}{\mathrm{d}q'}\eta(p-q-q')\right]; \tag{A 9}$$

We note finally that the singularity at $(q, q') = 0$ is removable when $\eta(p)$ is a smooth function for every p.

Appendix B

The functions F_k as defined in (44) are given explicitly by

$$F_0 = 1;\quad F_1 = \eta_x;\quad F_2 = \eta_x^2+\eta\eta_{xx}; \qquad \text{(B 1), (B 2), (B 3)}$$

$$F_3 = \eta_x^3+4\eta\eta_x\,\eta_{xx}+\eta^2\eta_{xxx}; \tag{B 4}$$

$$F_4 = \eta_x^4+(11\eta_x^2+4\eta\eta_{xx})\,\eta\eta_{xx}+7\eta^2\eta_x\,\eta_{xxx}+\eta^3\eta_{xxxx}; \tag{B 5}$$

On substituting in (46) we obtain

$$\epsilon = -\tfrac{1}{3}(\eta_x^2+\eta\eta_{xx}) - \tfrac{1}{45}[-9\eta_x^4-6(4\eta_x^2+\eta\eta_{xx})\,\eta\eta_{xx}+2\eta^2\eta_x\,\eta_{xxx}+\eta^3\eta_{xxxx}]\ldots. \tag{B 6}$$

This result can also be derived from (30) and (33) by iteration.

Appendix C

To determine the integrand $1/(1+\epsilon)$ in (50) from the iteration scheme (41) in the case of third-order Stokes waves in deep water, we start with the expansion

$$\eta = h+\sum_{l=1}^{\infty} a_l \cos l\theta, \tag{C 1}$$

and likewise

$$\epsilon = \sum_{m=0}^{\infty} \beta_m \cos m\theta, \tag{C 2}$$

with

$$\theta = \kappa hp-\omega t, \quad p = x/h+O(a/h), \tag{C 3}$$

according to the definition of p in (32).

Substituting the expansions (C 1) and (C 2) into the expressions (42) for I_0, I_1 and I_2 we encounter integrals of the form

$$I_s(A) = \int_0^\infty \frac{\mathrm{d}q}{\exp \pi q-1}\sin Aq, \tag{C 4}$$

$$I_c(A,B) = \int_0^\infty\int_0^\infty \frac{\mathrm{d}q\,\mathrm{d}q'}{\exp \pi(q+q')-1}\cos(Aq+Bq'), \tag{C 5}$$

$$I_s(A,B,C) = \int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{d}q\,\mathrm{d}q'\,\mathrm{d}q''}{\exp \pi(q+q'+q'')-1}\sin(Aq+Bq'+Cq''). \tag{C 6}$$

Applying integration by parts repeatedly we can reduce the double and triple integrals (C 5) and (C 6) to the single integral (C 4):

$$I_s(A) = \tfrac{1}{2}(\coth A - 1/A), \tag{C 7}$$

$$I_c(A,B) = \int_0^\infty \frac{dq}{\exp \pi q - 1} \frac{\sin Bq - \sin Aq}{B-A}, \tag{C 8}$$

$$I_s(A,B,C) = \int_0^\infty \frac{dq}{\exp \pi q - 1} \frac{(C-B)\sin Aq + (B-A)\sin Cq + (A-C)\sin Bq}{(C-B)(B-A)(A-C)}. \tag{C 9}$$

Further we can derive from (C 8) and (C 9) the special cases

$$I_c(A,A) = \int_0^\infty \frac{dq}{\exp \pi q - 1} q \cos Aq, \tag{C 10}$$

and

$$I_s(A,A,A) = \frac{1}{2}\int_0^\infty \frac{dq}{\exp \pi q - 1} q^2 \sin Aq. \tag{C 11}$$

Now we are able to evaluate the integrals I_0, I_1 and I_2. Using the addition formulae and the rules for products of sines and cosines, we obtain, in the limit of $\kappa h \rightarrow \infty$:

$$I_0 = \sum_{l=1}^{\infty} l\kappa a_l \cos l\theta, \tag{C 12}$$

$$I_1 = \sum_{l=1}^{\infty} \sum_{m>l}^{\infty} l\kappa a_l \beta_m \{(m-l)/m\} \cos (m-l)\theta, \tag{C 13}$$

$$\begin{aligned} I_2 = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tfrac{1}{2} l\kappa a_l \beta_m \beta_n \{ &(m-l)(m+n-l)\, g(m,n) \cos (m+n-l)\theta \\ &+ (m-l)(m-n-l)\, g(m,-n) \cos (m-n-l)\theta \\ &+ (-m-l)(-m+n-l)\, g(-m,n) \cos(-m+n-l)\theta\}, \end{aligned} \tag{C 14}$$

with

$$\begin{aligned} g(\mu,\nu) &= 1/\mu\nu && \text{for } \mu+\nu < l < \mu, \\ &= -1/\nu(\mu+\nu) && \text{for } \mu < l < \mu+\nu, \\ &= 1/\mu(\mu+\nu) && \text{for } l < \min(\mu, \mu+\nu) \\ &= 0 && \text{for } l > \max(\mu, \mu+\nu) \end{aligned}$$

and l, μ, ν integers, $l > 0$.

We then have, to the fourth order in κa_1,

$$\begin{aligned} 1/(1+\epsilon) = 1 - I_0 + I_1 - I_2 = {} & 1 - \kappa a_1(1 - \tfrac{1}{2}\beta_2)\cos\theta \\ & -2(\kappa a_2 - \tfrac{1}{3}\kappa a_1 \beta_3 + \kappa a_1 \beta_1 \beta_2/12)\cos 2\theta - 3\kappa a_3 \cos 3\theta. \end{aligned} \tag{C 15}$$

From (41) and (C 2) we can find β_1, β_2 and β_3 to the desired order:

$$\beta_1 = \kappa a_1; \quad \beta_2 = 2\kappa a_2 + \tfrac{1}{2}\kappa^2 a_1^2; \quad \beta_3 = 3\kappa a_3 + 2\kappa^2 a_1 a_2 + \tfrac{1}{4}\kappa^3 a_1^3. \tag{C 16}$$

Substituting (C 16) into (C 15) then yields the expression (55).

Appendix D

We consider the integrals

$$C_0(\lambda) = \int_0^\infty dv \ln \tanh (v) \cos \lambda v, \tag{D 1}$$

$$C_n(\lambda) = \int_0^\infty dv \left(\frac{\sin v}{v}\right)^n \cos \lambda v, \quad n \geqslant 1. \tag{D 2}$$

The first integral can be evaluated using integration by parts, to yield

$$\left.\begin{aligned} C_0(\lambda) &= -\frac{\pi}{2\lambda}\tanh\frac{\pi}{4}\lambda \quad \text{for} \quad \lambda \neq 0, \\ C_0{}'(0) &= -\tfrac{1}{8}\pi^2. \end{aligned}\right\} \tag{D 3}$$

The remaining integrals can be evaluated successively, using integration by parts and the rules for products of sines and cosines (see also Gradshteyn & Ryzhik 1980; sec. 3.836 for integer values of $\lambda > 0$):

$$C_1(\lambda) = 0 \quad \text{for} \quad \lambda > 1, \qquad \tfrac{1}{4}\pi \quad \text{for} \quad \lambda = 1, \qquad \tfrac{1}{2}\pi \quad \text{for} \quad 0 \leqslant \lambda < 1; \tag{D 4}$$

$$C_2(\lambda) = 0 \quad \text{for} \quad \lambda \geqslant 2, \qquad \tfrac{1}{4}\pi(2-\lambda) \quad \text{for} \quad 0 \leqslant \lambda < 2; \tag{D 5}$$

$$C_3(\lambda) = 0 \quad \text{for} \quad \lambda \geqslant 3, \qquad \tfrac{1}{16}\pi(3-\lambda)^2 \quad \text{for} \quad 1 \leqslant \lambda < 3, \qquad \tfrac{1}{8}\pi(3-\lambda^2) \quad \text{for} \quad 0 \leqslant \lambda < 1; \tag{D 6}$$

$$C_4(\lambda) = 0 \quad \text{for} \quad \lambda \geqslant 4, \qquad \tfrac{1}{96}\pi(4-\lambda)^3 \quad \text{for} \quad 2 \leqslant \lambda < 4, \qquad \tfrac{1}{96}\pi(32-12\lambda^2+3\lambda^3) \quad \text{for} \quad 0 \leqslant \lambda < 2; \tag{D 7}$$

Finally we note that these integrals are normalized as follows:

$$\int_0^\infty C_n(\lambda)\,\mathrm{d}\lambda = \tfrac{1}{2}\pi \quad \text{for all} \quad n \geqslant 1. \tag{D 8}$$

Appendix E

According to the definition of the variational derivative, as found for example in Broer & Kobussen (1972), we have from (71) and (51) for the derivative with respect to ϕ,

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{\delta \mathscr{H}}{\delta \phi} f = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x' [\phi_x f_{x'} + \phi_{x'} f_x] R_0(x, x'; \eta), \tag{E 1}$$

for *all* functions $f(x)$ of some class $C\{f\}$. Applying integration by parts and using the symmetry of the kernel R_0, we find

$$\frac{\delta \mathscr{H}}{\delta \phi} = -\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}x' \, \phi_{x'} R_0(x, x'; \eta). \tag{E 2}$$

(Note that (E 2) can be derived in a more simple way from (E 1) by using the definition of the *derivative* of the delta function.) To obtain the variational derivative with respect to ζ, we proceed as follows. According to the definition,

$$\int_{-\infty}^{\infty} \mathrm{d}x' \frac{\delta \mathscr{H}}{\delta \zeta} f = \int_{-\infty}^{\infty} \mathrm{d}x' g \zeta f + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \frac{\phi_{x'}\phi_{x''}}{2\sinh \tfrac{1}{2}\pi \left| \int_{x'}^{x''} \mathrm{d}r/\eta \right|} \left| \int_{x'}^{x''} \mathrm{d}r f/\eta^2 \right|. \tag{E 3}$$

As this equation must hold for *all* functions $f(x) \in C$, we may take any sequence $f_n(x)$ approaching the delta function $\delta(x)$, in (E 3).

Denoting the unit function of Heaviside by $U(x)$, we have

$$\int_{x'}^{x''} \mathrm{d}r\, \delta(r-x)/\eta^2(r) = [U(x-x')\,U(x''-x)]/\eta^2(x). \tag{E 4}$$

Hence

$$\frac{\delta \mathscr{H}}{\delta \zeta} = g\zeta + \frac{1}{2\eta^2}\int_{-\infty}^{\infty} \mathrm{d}x'\,\phi_{x'}\left[\int_{-\infty}^{x'} \frac{\phi_{x''}\,U(x-x'')\,U(x'-x)\,\mathrm{d}x''}{2\sinh\frac{1}{2}\pi\left|\displaystyle\int_{x'}^{x''}\mathrm{d}r/\eta\right|}\right.$$

$$\left.+\int_{x'}^{\infty} \frac{\phi_{x''}\,U(x-x')\,U(x''-x)\,\mathrm{d}x''}{2\sinh\frac{1}{2}\pi\left|\displaystyle\int_{x'}^{x''}\mathrm{d}r/\eta\right|}\right] = g\zeta + \frac{1}{2\eta^2}\int_{-\infty}^{x}\mathrm{d}x'\int_{x}^{\infty}\mathrm{d}x''\frac{\phi_{x'}\,\phi_{x''}}{\sinh\frac{1}{2}\pi\left|\displaystyle\int_{x'}^{x''}\mathrm{d}r/\eta\right|}. \quad \text{(E 5)}$$

From (E 2) and (E 5) the canonical equations (72) follow immediately.

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